

THE HÖLDER CONTINUITY OF SPECTRAL MEASURES OF AN EXTENDED CMV MATRIX

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ABSTRACT. We prove results about the Hölder continuity of the spectral measures of the extended CMV matrix, given power law bounds of the solution of the eigenvalue equation. We thus arrive at a unitary analogue of the results of Damanik, Killip and Lenz about the spectral measure of the discrete Schrödinger operator.

1. INTRODUCTION

The CMV matrix is a central object in the study of orthogonal polynomials on the unit circle (OPUC), where it plays a role analogous to that of the Jacobi matrix in the study of orthogonal polynomials on the real line (OPRL). More precisely, given a probability measure on the unit circle, we can perform a Gram-Schmidt orthogonalization process on $\{1, z, z^2, \dots\}$ using the standard L^2 inner product to obtain a sequence of orthogonal polynomials $\{\varphi_0(z), \varphi_1(z), \varphi_2(z), \dots\}$ that obey a recursion relation known as the Szegő recursion, given by $\varphi_0(z) = 1$ and

$$\varphi_{n+1}(z) = \frac{z\varphi_n(z) - \overline{\alpha(n)}\varphi_n^{*,n}(z)}{\rho(n)}.$$

Here $\{\alpha(n)\}_{n=0}^\infty \in \mathbb{D}^\infty$ are the recursion coefficients (also known as *Verblunsky coefficients*), and $\rho(n) = \sqrt{1 - |\alpha(n)|^2}$. The *,n operator is defined by $P^{*,n}(z) = z^n \overline{P(1/\bar{z})}$.

The Szegő recursion can be written in a matrix form as follows:

$$\begin{pmatrix} \varphi_{n+1}(z) \\ \varphi_{n+1}^*(z) \end{pmatrix} = \frac{1}{\rho(n)} \begin{pmatrix} z & -\overline{\alpha(n)} \\ -\alpha(n)z & 1 \end{pmatrix} \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix}. \quad (1)$$

The (one-sided) CMV matrix \mathcal{C} is a unitary operator on $\ell^2(\mathbb{Z}_{\geq 0})$ given by

$$\begin{pmatrix} \overline{\alpha(0)} & \overline{\alpha(1)}\rho(0) & \rho(1)\rho(0) & 0 & 0 & \dots \\ \rho(0) & -\overline{\alpha(1)}\alpha(0) & -\overline{\rho(1)}\alpha(0) & 0 & 0 & \dots \\ 0 & \overline{\alpha(2)}\rho(1) & -\overline{\alpha(2)}\alpha(1) & \overline{\alpha(3)}\rho(2) & \rho(3)\rho(2) & \dots \\ 0 & \rho(2)\rho(1) & -\rho(2)\alpha(1) & -\overline{\alpha(3)}\alpha(2) & -\overline{\rho(3)}\alpha(2) & \dots \\ 0 & 0 & 0 & \overline{\alpha(4)}\rho(3) & -\overline{\alpha(4)}\alpha(3) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (2)$$

There are numerous important connections between the matrix, the polynomials, and the probability measure on the unit circle. For instance, the spectral measure of \mathcal{C} is exactly the probability measure from which the orthogonal polynomials are derived. The CMV matrix thus enables us to use spectral theoretic tools in OPUC, and enables us to understand OPUC as a unitary analogue of OPRL. We recommend [Sim04] as a good reference for OPUC and the CMV matrix.

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Now let us redefine $\{\alpha(n)\}_{n=-\infty}^{\infty}$ as a two-sided infinite sequence, so we can introduce the two-sided CMV matrix \mathcal{E} ,

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\overline{\alpha(0)}\alpha(-1) & \overline{\alpha(1)}\rho(0) & \rho(1)\rho(0) & 0 & 0 & \cdots \\ \cdots & -\rho(0)\alpha(-1) & -\overline{\alpha(1)}\alpha(0) & -\overline{\rho(1)}\alpha(0) & 0 & 0 & \cdots \\ \cdots & 0 & \overline{\alpha(2)}\rho(1) & -\overline{\alpha(2)}\alpha(1) & \overline{\alpha(3)}\rho(2) & \rho(3)\rho(2) & \cdots \\ \cdots & 0 & \rho(2)\rho(1) & -\rho(2)\alpha(1) & -\overline{\alpha(3)}\alpha(2) & -\overline{\rho(3)}\alpha(2) & \cdots \\ \cdots & 0 & 0 & 0 & \overline{\alpha(4)}\rho(3) & -\overline{\alpha(4)}\alpha(3) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \quad (3)$$

Here \mathcal{E} is a unitary operator on $\ell^2(\mathbb{Z})$. While \mathcal{E} has a somewhat looser connection with the orthogonal polynomials since the set of recursion coefficients is now two-sided infinite, it is the more natural object of study when we choose to define the $\alpha(n)$ dynamically (please refer to sections 10.5-10.16 of [Sim04] for a background on this point of view).

Furthermore, the two-sided CMV matrix has emerged as a useful tool in the study of quantum walks, a connection first explored in [CGMV10]. For certain models of quantum walks, where $\psi \in \ell^2(\mathbb{Z})$ refers to the initial state of a particle, $\mathcal{E}^k \psi$ gives us the state after a quantum walk of k steps.

There is, naturally, a way to relate the measures of an extended CMV matrix and the measures of the constituent halves. This formula was discovered by Gesztesy and Zinchenko, and will be explained in the next section. As an application of this formula we present an OPUC analogue of [DKL00]. That is, we derive theorems about the Hölder continuity of the spectral measures of \mathcal{E} given power law bounds on the entries of formal eigenvectors of one of the corresponding \mathcal{C} “halves”.

In [IRT92], [DL99], and [DKL00] power law bounds on formal eigenvectors are established for OPRL with a Sturmian sequence of recursion coefficients. By *Sturmian*, we mean that $\alpha(n) = v(n)\alpha + (1 - v(n))\beta$, where α, β (the *alphabet*) are complex numbers of modulus less than one, and $v(n) = [(n+1)\omega] - [n\omega]$. Here $\omega \in (0, 1) \setminus \mathbb{Q}$ is the *frequency*. In Section 4, we apply analogous methods to extended CMV matrices with ω equal to the golden mean.

This is the second of a trilogy of loosely related papers concerned with the dynamical spreading of the time-independent quantum walk model whose coins arise from a Fibonacci sequence. More precisely, the coin at the n th site is rotation by θ_n , where θ is any suffix of a Fibonacci sequence (see [Sim04] 12.8):

$$\dots \theta_a, \theta_b, \theta_a, \theta_a, \theta_b, \dots$$

The first of the trilogy is [DMY13], which establishes that the formal solutions u of $\mathcal{E}u = zu$ obey certain power law bounds (where \mathcal{E} is the transition matrix that describes the quantum walk above). This paper shows that u obeying power law bounds implies Hölder continuity of spectral measures of \mathcal{E} . Lastly, [DFV] asserts that for unitary operators, Hölder continuity of spectral measure implies dynamical spreading of the corresponding quantum walk.

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2. MAIN TOOLS AND BACKGROUND

In this section, we will explain an essential formula proven in [GZ06]. We require a way to relate the extended CMV operator \mathcal{E} with the two one-sided CMV operators that comprise its two halves. More precisely, if we modify $\alpha(-1) = -1$, then (3) becomes the direct sum of operators on $\ell^2([0, \infty) \cap \mathbb{Z})$ and $\ell^2([-1, -\infty) \cap \mathbb{Z})$ of

the form (2). We label the halves as \mathcal{C}_+ and \mathcal{C}_- respectively. We also note that henceforth in this paper \mathcal{E} refers to the unmodified extended CMV matrix.

First, let us label $F_+(z)$, the Carathéodory function corresponding to \mathcal{C}_+ , and $F_-(z)$, the Carathéodory function corresponding to \mathcal{C}_- . Carathéodory functions are holomorphic maps from \mathbb{D} to the right half plane $\{z | \operatorname{Re} z > 0\}$. We also say a function is anti-Carathéodory when its negative is Carathéodory. The correspondences between a given CMV matrix and its Carathéodory function are explored more fully in Section 1.3 of [Sim04]. Briefly, a Carathéodory function is the CMV analogue of the m -function in the theory of Jacobi matrices, and is connected to the spectral theory of the CMV matrix.

For example, where c_i are the moments of the spectral measure of the one-sided CMV matrix \mathcal{C} , its Carathéodory function F may be expressed as $F(z) = 1 + 2 \sum_{n=1}^{\infty} c_n z^n$. It is also true that $\operatorname{Re} F(re^{i\theta}) d\theta / 2\pi$ converges weakly to the spectral measure of \mathcal{C} as $r \rightarrow 1$. Finally, we note that where μ is the spectral measure of a CMV matrix, its Carathéodory function is given by the formula

$$F(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).$$

The Green's function (or resolvent function) for \mathcal{E} is computed using formal eigenvalues to \mathcal{C}_{\pm} and \mathcal{C}_{\pm}^T .

Lemma 1 (Lemma 3.1 in [GZ06]). *Let $z \in \mathbb{C} \setminus (\partial\mathbb{D} \cup \{0\})$, and let M_- be an anti-Carathéodory function in ([GZ06], Lemma 2.20), which is, by (2.139) in [GZ06] related to F_- by*

$$M_-(z) = \frac{\operatorname{Re}(1 - \overline{\alpha}_0) - i\operatorname{Im}(1 + \overline{\alpha}_0)F_-(z)}{i\operatorname{Im}(1 - \overline{\alpha}_0) - \operatorname{Re}(1 + \overline{\alpha}_0)F_-(z)}.$$

Let u_{\pm} be ℓ^2 solutions to $(\mathcal{C}_{\pm} - z)u = 0$, and let v_{\pm} be ℓ^2 solutions to $(\mathcal{C}_{\pm}^T - z)v = 0$, normalized by

$$\begin{aligned} v_-(z, 0) &= -1 + M_-(z), \quad v_+(z, 0) = -1 + F_+(z), \\ u_-(z, 0) &= z + zM_-(z), \quad u_+(z, 0) = z + zF_+(z). \end{aligned}$$

We may extend these solutions to solutions of $(\mathcal{E} - z)w = 0$ and $(\mathcal{E}^T - z)w = 0$.

Then the resolvent function $(\mathcal{E} - z)^{-1}(x, y)$ can be expressed as

$$\frac{-1}{2z(F_+(z) - M_-(z))} \begin{cases} u_-(z, x)v_+(z, y) & \text{if } x < y \text{ or } x = y \text{ odd,} \\ u_+(z, x)v_-(z, y) & \text{if } x > y \text{ or } x = y \text{ even,} \end{cases} \quad (4)$$

3. APPLICATIONS OF THE FORMULA

For a $\ell^2(\mathbb{N})$ -vector u , and a positive integer n , we define $\|u\|_n$ as $\sqrt{\sum_{j=0}^n |u(j)|^2}$. We can also define $\|u\|_k$, for k positive but not an integer as a linear interpolation of $\|u\|_n$.

Lemma 2. *Suppose, for a one-sided CMV matrix \mathcal{C} , that every solution of*

$$\begin{pmatrix} \eta_{n+2}(z) \\ \eta_{n+1}(z) \end{pmatrix} = \frac{1}{\rho(n)} \begin{pmatrix} z & -\overline{\alpha(n)} \\ -\alpha(n)z & 1 \end{pmatrix} \begin{pmatrix} \eta_n(z) \\ \eta_{n-1}(z) \end{pmatrix}, \quad (5)$$

with $|\eta_0(z)|^2 + |\eta_1(z)|^2 = 2$ obeys the estimate

$$C_1 L^{\gamma_1} \leq \|\eta(z)\|_L \leq C_2 L^{\gamma_2}, \quad (6)$$

for $L > 0$ sufficiently large. Then

$$\sup_{\lambda \in \partial\mathbb{D}} \left| \frac{(1 - \lambda) + (1 + \lambda)F(rz)}{(1 + \lambda) + (1 - \lambda)F(rz)} \right| \leq C_3 (1 - r)^{\beta-1}, \quad (7)$$

where $\beta = 2\gamma_1/(\gamma_1 + \gamma_2)$.

Proof. This is a consequence of the Jitomirskaya-Last inequality for OPUC (see [Sim04] Section 10.8), which says that

$$\frac{\|\psi^\lambda(z)\|_{x(r)}}{\|\varphi^\lambda(z)\|_{x(r)}} \lesssim |F^\lambda(rz)| \lesssim \frac{\|\psi^\lambda(z)\|_{x(r)}}{\|\varphi^\lambda(z)\|_{x(r)}}.$$

Here, F^λ is the Carathéodory function corresponding to the Alexandrov measure μ_λ (refer to Theorem 3.2.14 of [Sim04]). Its first and second kind orthogonal polynomials are φ^λ and ψ^λ . The function $x(r)$ is defined by $(1-r)\|\varphi^\lambda(z)\|_{x(r)}\|\psi^\lambda(z)\|_{x(r)} = \sqrt{2}$.

The required inequality is equivalent to $|F^\lambda(rz)| \lesssim (1-r)^{\beta-1}$. This is true if $\frac{\|\psi^\lambda(z)\|_{x(r)}}{\|\varphi^\lambda(z)\|_{x(r)}} \lesssim (1-r)^{\beta-1}$, by the Jitomirskaya-Last inequality. Because φ^λ and ψ^λ solve $(\eta_{n+2}(z), \eta_{n+1}(z))^T = T_n(z)(\eta_n(z), \eta_{n-1}(z))^T$ with initial conditions $(1, \bar{\lambda})$ and $(1, -\bar{\lambda})$, the hypothesis applies to ψ^λ and φ^λ . Therefore,

$$\|\psi^\lambda(z)\|_{x(r)}^\beta \|\varphi^\lambda(z)\|_{x(r)}^{\beta-2} \lesssim x(r)^{\gamma_1(\beta-2)+\gamma_2\beta} \simeq 1.$$

By the definition of $x(r)$, it follows that

$$\|\psi^\lambda(z)\|_{x(r)}^\beta \|\varphi^\lambda(z)\|_{x(r)}^{\beta-2} \lesssim (1-r)^{\beta-1} \|\psi^\lambda(z)\|_{x(r)}^{\beta-1} \|\varphi^\lambda(z)\|_{x(r)}^{\beta-1},$$

which is equivalent to the required inequality. \square

Theorem 1. *Given $z \in \Sigma$, suppose that the estimate (6) holds. Then, where $G_{kl}(z) = (\delta_k, (\mathcal{E} - z)^{-1}\delta_l)$,*

$$|G_{00}(rz) + G_{11}(rz)| \leq C_4(1-r)^{\beta-1},$$

for all $r \in (0.9, 1)$ and C_4 a z and r -independent constant. Consequently, Λ is β -Hölder continuous at z .

In particular, assume that $S \subset \partial\mathbb{D}$ is a Borel set such that there are constants γ_1, γ_2 and, for each $z \in S$, there are constants $C_1(z), C_2(z)$ so that

$$C_1(z)L^{\gamma_1} \leq \|\eta\|_L \leq C_2(z)L^{\gamma_2}$$

for every $z \in S$ and for every solution of (5) that is normalized. Then, the restriction of every spectral measure of \mathcal{E} to S is purely $\frac{2\gamma_1}{\gamma_1+\gamma_2}$ -continuous, that is, it gives zero weight to sets of zero $h^{\frac{2\gamma_1}{\gamma_1+\gamma_2}}$ measure.

Proof. The following is a maximum modulus principle argument similar to that in [DKL00]. Fix $z \in \Sigma$ and $r \in (0.9, 1)$. We consider (7) and obtain

$$\sup_{\lambda \in \partial\mathbb{D}} \left| \frac{(1-\lambda) + (1+\lambda)F_+(rz)}{(1+\lambda) + (1-\lambda)F_+(rz)} \right| \leq C_3(1-r)^{\beta-1}. \quad (8)$$

Since $-M_-(z)$ is a Carathéodory function it maps to the right half plane, and so the expression $(M_-(rz) + 1)/(M_-(rz) - 1)$ has modulus less than 1. Thus by the maximum modulus principle,

$$\left| \frac{\left(1 - \frac{M_-(rz)+1}{M_-(rz)-1}\right) + \left(1 + \frac{M_-(rz)+1}{M_-(rz)-1}\right) F_+(rz)}{\left(1 + \frac{M_-(rz)+1}{M_-(rz)-1}\right) + \left(1 - \frac{M_-(rz)+1}{M_-(rz)-1}\right) F_+(rz)} \right| \leq C_3(1-r)^{\beta-1}.$$

Now if we simplify the expression on the left, we have

$$\left| \frac{1 - M_-(rz)F_+(rz)}{F_+(rz) - M_-(rz)} \right|.$$

A table on page 181 of [GZ06] computes the values of $u_{\pm}(1)$, $u_{\pm}(0)$, $v_{\pm}(1)$, and $v_{\pm}(0)$. We use $k_0 = -1$ in that table. Note that we index our solutions u, v differently, so in their notation,

$$\begin{aligned} u_+(n) &= q_+(n-1) + F_+ p_+(n-1), \\ u_-(n) &= q_+(n-1) + M_- p_+(n-1), \\ v_+(n) &= s_+(n-1) + F_+ r_+(n-1), \\ v_-(n) &= s_+(n-1) + M_- r_+(n-1). \end{aligned}$$

We also label our Verblunsky coefficients differently than they do: their α_n is written as $-\overline{\alpha(n)}$ in our notation. Using these calculations and (4), we can write $G_{00} + G_{11}$ as

$$-\frac{(-1 + F_+)(1 + M_-)}{2(F_+ - M_-)} - \frac{[z + \overline{\alpha(0)} + M_-(z - \overline{\alpha(0)})][-1 - \alpha(0)z + F_+(1 - \alpha(0)z)]}{2\rho(0)^2 z(F_+ - M_-)}.$$

For r approaching 1, $|G_{00}(rz) + G_{11}(rz)|$ gets large when $F_+(rz) - M_-(rz)$ is close to zero, or when $F_+(rz)$ and $F_-(rz)$ both go to infinity. In both these cases,

$$|G_{00}(rz) + G_{11}(rz)| \leq C_4 \left| \frac{1 - M_-(rz)F_+(rz)}{F_+(rz) - M_-(rz)} \right|,$$

for an appropriate constant C_4 . It is not difficult to see then, as a consequence

$$|G_{00}(rz) + G_{11}(rz)| \leq C_4(1-r)^{\beta-1}.$$

Let us first note the connection between $G_{00} + G_{11}$ and the Carathéodory function F corresponding to \mathcal{E} and $d\Lambda$. We have by definition

$$F(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\Lambda(\theta).$$

Let us also define

$$d\Lambda_r(\theta) = \operatorname{Re} F(re^{i\theta}) \frac{d\theta}{2\pi}.$$

It is well known that $d\Lambda_r$ converges to $d\Lambda$ weakly. We note also that

$$\begin{aligned} F(z) &= \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\Lambda(\theta) \\ &= 1 + 2z \int \frac{1}{e^{i\theta} - z} d\Lambda(\theta) \\ &= 1 + 2z(G_{00}(z) + G_{11}(z)). \end{aligned}$$

We then deduce that $\Lambda(z)$ is uniformly β -Hölder continuous on Σ . Writing $z = e^{i\Theta}$:

$$\Lambda[e^{i(\Theta-\epsilon)}, e^{i(\Theta+\epsilon)}] = \int_{\Theta-\epsilon}^{\Theta+\epsilon} 1 d\Lambda(\Theta).$$

We note that for sufficiently small ϵ the above is less than

$$2\epsilon (\operatorname{Re} F((1-\epsilon)z) + 1) \leq C\epsilon^\beta,$$

since $\beta \leq 1$. □

Note that if we let C_1, C_2 be z -dependent, the theorem still holds, except that C_3 is also z -dependent. We can then conclude:

Theorem 2. *Let Σ be a Borel subset of $\partial\mathbb{D}$, and let \mathcal{C} be a CMV operator on $\ell^2(\mathbb{N})$. Suppose there are constants γ_1, γ_2 such that for each $z \in \Sigma$, every normalized solution of $\eta(z)$ of the transfer matrix recursion (5) obeys the estimate*

$$C_1(z)L^{\gamma_1} \leq \|\eta\|_L \leq C_2(z)L^{\gamma_2}$$

for $L > 0$ sufficiently large. Let $\beta = 2\gamma_1/(\gamma_1 + \gamma_2)$. Then any extension \mathcal{E} of \mathcal{C} to $\ell^2(\mathbb{Z})$ has purely β -continuous spectrum on Σ . Moreover, if $C_1(z)$ and $C_2(z)$ are independent of z , then for any $\varphi \in \ell^2$ of compact support, the spectral measure of (\mathcal{E}, φ) is uniformly β -Hölder continuous on Σ .

Before we proceed, we state and prove the following well-known fact for the reader's convenience:

Lemma 3. *For any $n \in \mathbb{Z}$, $\{\delta_{2n}, \delta_{2n+1}\}$ form a spectral basis for \mathcal{E} .*

Proof of lemma. First, let us show that δ_{2n+2} is in the span $S_{2n,2n+1}$ of

$$\{\mathcal{E}^k \delta_{2n}\}_{k \in \mathbb{Z}} \cup \{\mathcal{E}^k \delta_{2n+1}\}_{k \in \mathbb{Z}}.$$

First, note that we have

$$\mathcal{E}\delta_{2n+1} = \overline{\alpha(2n+1)}\rho(2n)\delta_{2n} - \overline{\alpha(2n+1)}\alpha(2n)\delta_{2n+1} \quad (9)$$

$$+ \overline{\alpha(2n+2)}\rho(2n+1)\delta_{2n+2} + \rho(2n+2)\rho(2n+1)\delta_{2n+3},$$

$$\mathcal{E}\delta_{2n+2} = \rho(2n+1)\rho(2n)\delta_{2n} - \rho(2n+1)\alpha(2n)\delta_{2n+1} \quad (10)$$

$$- \overline{\alpha(2n+2)}\alpha(2n+1)\delta_{2n+2} - \rho(2n+2)\alpha(2n+1)\delta_{2n+3}.$$

This gives us

$$\begin{aligned} \frac{\alpha(2n+1)}{\rho(2n+1)}\mathcal{E}\delta_{2n+1} + \mathcal{E}\delta_{2n+2} &= \left(\frac{|\alpha(2n+1)|^2\rho(2n)}{\rho(2n+1)} + \rho(2n+1)\rho(2n) \right) \delta_{2n} \\ &\quad - \left(\frac{|\alpha(2n+1)|^2\alpha(2n)}{\rho(2n+1)} + \rho(2n+1)\alpha(2n) \right) \delta_{2n+1}, \end{aligned}$$

and we conclude that $S_{2n,2n+1}$ contains $\mathcal{E}\delta_{2n+2}$. Applying \mathcal{E}^{-1} on both sides of the preceding equation shows that it also contains δ_{2n+2} .

By considering the expressions for $\mathcal{E}\delta_{2n-1}$ and $\mathcal{E}\delta_{2n}$ instead, we can similarly show that $\mathcal{E}\delta_{2n-1}$, and hence δ_{2n-1} lies in $S_{2n,2n+1}$.

Now let us demonstrate that δ_{2n+3} is in $S_{2n,2n+1}$. We consider (9) and (10) once more, and this time by eliminating the $\delta_{2n}, \delta_{2n+1}$ terms we get

This gives us

$$\begin{aligned} \mathcal{E}\delta_{2n+1} - \frac{\overline{\alpha(2n+1)}}{\rho(2n+1)}\mathcal{E}\delta_{2n+2} &= \left(\overline{\alpha(2n+2)}\rho(2n+1) + \frac{|\alpha(2n+1)|^2\alpha(2n+1)}{\rho(2n+1)} \right) \delta_{2n+2} \\ &\quad + \left(\rho(2n+2)\rho(2n+1) + \frac{|\alpha(2n+1)|^2\rho(2n+2)}{\rho(2n+1)} \right) \delta_{2n+3}, \end{aligned}$$

and this demonstrates that δ_{2n+3} lies in $S_{2n+1,2n+2}$, and hence $S_{2n,2n+1}$.

We can similarly show that δ_{2n-2} lies in $S_{2n-1,2n}$ and hence $S_{2n,2n+1}$, by using the expressions for $\mathcal{E}\delta_{2n-1}$ and $\mathcal{E}\delta_{2n}$ and then eliminating the $\delta_{2n}, \delta_{2n+1}$ terms.

We have now shown that $S_{2n,2n+1}$ contains $\{\delta_{2n-2}, \delta_{2n-1}, \delta_{2n+2}, \delta_{2n+3}\}$. A simple induction argument now tells us that $S_{2n,2n+1} = \ell^2(\mathbb{Z})$. \square

Remark. It is easy to see that $\{\delta_{2n-1}, \delta_{2n}\}$ for any n also form a spectral basis.

Proof of Theorem 2. From the lemma and its proof, we see that given a $\phi \in \ell^2(\mathbb{Z})$ with support on $\{-N, \dots, N+1\}$, there must exist polynomials P_0, P_1 of degree not exceeding N such that $P_0(\mathcal{E})\delta_0 + P_1(\mathcal{E})\delta_1 = \phi$. This implies that the spectral measure for ϕ is bounded by $q(z)d\Lambda(z)$ for some polynomially bounded function $q(z)$. If C_1, C_2 are independent of z , then, by the corollary $d\Lambda$ is uniformly β -Hölder continuous, and this implies that $qd\Lambda$ is also uniformly β -Hölder continuous. In the case that C_1, C_2 depend on z , we know that $d\Lambda$ is β -continuous. Given any $\phi \in \ell^2$, its spectral measure is dominated by Λ and so must be β -continuous as well. \square

4. STURMIAN VERBLUNSKY COEFFICIENTS

Theorem 3. *Given a sequence $\{A_n : \mathbb{T} \rightarrow \text{SL}(2, \mathbb{C})\}_{n=0}^\infty$, let us write $M_k(z) = \prod_{n=k}^0 A_n(z)$. Suppose there are sequences a_n and q_n of natural numbers related by $q_{n+1} = a_{n+1}q_n + q_{n-1}$, such that $M_{q_{n+1}}(z) = M_{q_{n-1}}(z)M_{q_n}(z)^{a_{n+1}}$. Let $x_n(z) = \text{tr} M_{q_n}(z)$ and $z_n(z) = \text{tr} M_{q_{n-1}}(z)M_{q_n}(z)$, and put $I(z) = x_{n-1}^2 + x_n^2 + z_n^2 - x_{n-1}x_nz_n$.*

Suppose that:

- (1) *The function I is independent of n .*
- (2) *The sequence a_n is of bounded density: $d = \limsup \frac{1}{N} \sum_{n=1}^N a_n$ is finite.*
- (3) *There is a compact set $\Sigma \subset \mathbb{T}$ and a constant K such that $z \in \Sigma$ iff $|x_n(z)| \leq K$ or $|z_n(z)| \leq K$ for all n .*

Then for all $z \in \Sigma$, there exist $\gamma_2(z)$ and $C(z)$ independent of n such that

$$\|M_n(z)\| \leq C(z)n^{\gamma_2(z)}.$$

These conditions are sufficient to apply the argument in [IRT92]. One obtains

$$C(z) = L^{4d}, \quad \gamma_2(z) = 4d \log_2 L,$$

where

$$L = \max(4 \max(2, \sup |x_n|, \sup |z_n|), 4\|M_1\|, 4\|M_0\|, 4\|M_0M_1\|) \\ \times (4 + 2 \max(2, \sup |x_n|, \sup |z_n|)).$$

The method used in [DKL00] can be applied to show that for some C' ,

$$\|\xi\|_L \leq C'(z)n^{\gamma(z)}$$

for any solution $\xi(z)$ to the transfer matrix recursion. Compactness of Σ and continuity of C, γ yield a z -independent bound by taking the maximum.

Theorem 4. *Let q_n be the convergents of the continued fraction $[a_1, a_2, a_3, \dots]$. Relaxing the bounded density hypothesis to require only that q_n be bounded above by a geometric sequence, such a sequence of maps into $\text{SL}(2, \mathbb{C})$ satisfies, for all $z \in \Sigma$,*

$$\|\xi(z)\|_L \geq C_2(z)L^{\gamma_1(z)}$$

for some $C(z), \gamma_1(z)$, and for L large enough.

Proof. The method used in [DKL00] applies without any significant changes. It only deals with model-independent properties of the transfer matrices. \square

Claim 1. *In both cases, an extended CMV matrix with Verblunsky coefficients that have a Fibonacci sequence as a suffix furnishes an example of such a sequence of maps.*

Proof. Let $\{T_n(z)\}_{n=0}^\infty$ be the sequence of n -step transfer matrices corresponding to the quasiperiodic CMV operator \mathcal{E} . Then $\det(T_n(z)) = z^n$, so that $M_n(z) := T_n(z)/z^{n/2}$ is in $\mathrm{SL}(2, \mathbb{C})$. It is well known (see [Sim04] 12.8) that the family T_n obeys a substitution rule of the necessary type; so does the family M_n . Because the spectrum of \mathcal{E} is contained in \mathbb{T} , M_n and T_n always have the same operator norms. Finally, that the traces $x_n(z)$ obey the required bound is proved in [Sim04] 12.8.

The method in [DKL00] provides a simple expression for γ_1 . Put $q_n \leq B^n$, and let $C(\alpha, \beta) := \max\{\max_{|z|=1} 2 + \sqrt{8 + I(z)}, \frac{4}{\sqrt{1-|\alpha|^2}\sqrt{1-|\beta|^2}}\}$. Then

$$\gamma_1 = \frac{\log\left(1 + \frac{1}{4C(\alpha, \beta)^2}\right)}{16 \log B}.$$

The constant $C(\alpha, \beta)$ occurs because it bounds $|x_n(z)|$ for $z \in \Sigma$. □

Corollary 1. *With the notation and assumptions above, the spectral measure of such a CMV operator is uniformly β -Hölder continuous for $\beta = \frac{2\gamma_1}{\gamma_1 + \gamma_2}$.*

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